Proposition 14.32 (Invariant Formula for the Exterior Derivative). *Let M be a smooth manifold with or without boundary, and $\omega \in \Omega^k(M)$. For any smooth vector fields $X_1, \dots, X_{k+1}$ on M,*

$$
\begin{aligned}
d\omega(X_1, \dots, X_{k+1}) &= \sum_{1 \le i \le k+1} (-1)^{i-1} X_i\left(\omega(X_1, \dots, \widehat{X_i}, \dots, X_{k+1})\right) \\
&\quad + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{k+1}), \qquad (14.29)
\end{aligned}
$$

where the hats indicate omitted arguments.

Proof. For this proof, let us denote the entire expression on the right-hand side of (14.29) by $D\omega(X_1, \dots, X_{k+1})$, and the two sums on the right-hand side by $I(X_1, \dots, X_{k+1})$ and $II(X_1, \dots, X_{k+1})$, respectively. Note that $D\omega$ is obviously multilinear over $\mathbb{R}$. We begin by showing that, like $d\omega$, it is actually multilinear over $C^\infty(M)$, which is to say that for $1 \le p \le k+1$ and $f \in C^\infty(M)$,

$$
D\omega(X_1, \dots, fX_p, \dots, X_{k+1}) = fD\omega(X_1, \dots, X_p, \dots, X_{k+1}).
$$

In the expansion of $I(X_1, \dots, fX_p, \dots, X_{k+1})$, f obviously factors out of the $i = p$ term. The other terms expand as follows:

$$
\begin{aligned}
&\sum_{i \ne p} (-1)^{i-1} X_i\left(f\omega(X_1, \dots, \widehat{X_i}, \dots, X_{k+1})\right) \\
&\quad = \sum_{i \ne p} (-1)^{i-1} \Big(fX_i\left(\omega(X_1, \dots, \widehat{X_i}, \dots, X_{k+1})\right) \\
&\qquad + (X_i f)\omega(X_1, \dots, \widehat{X_i}, \dots, X_{k+1})\Big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&I(X_1, \dots, fX_p, \dots, X_{k+1}) \\
&\quad = fI(X_1, \dots, X_p, \dots, X_{k+1}) + \sum_{i \ne p} (-1)^{i-1} (X_i f)\omega(X_1, \dots, \widehat{X_i}, \dots, X_{k+1}). \qquad (14.30)
\end{aligned}
$$

Consider next the expansion of II. Again, f factors out of all the terms in which $i \ne p$ and $j \ne p$. To expand the other terms, we use (8.11), which implies

$$
\begin{aligned}
[fX_p, X_j] &= f[X_p, X_j] - (X_j f)X_p, \\
[X_i, fX_p] &= f[X_i, X_p] + (X_i f)X_p.
\end{aligned}
$$

Inserting these formulas into the $i = p$ and $j = p$ terms, we obtain

$$
\begin{aligned}
&II(X_1, \dots, fX_p, \dots, X_{k+1}) \\
&\quad = fII(X_1, \dots, X_p, \dots, X_{k+1}) \\
&\qquad + \sum_{p<j} (-1)^{p+j+1} (X_j f)\omega(X_p, X_1, \dots, \widehat{X_p}, \dots, \widehat{X_j}, \dots, X_{k+1}) \\
&\qquad + \sum_{i<p} (-1)^{i+p} (X_i f)\omega(X_p, X_1, \dots, \widehat{X_i}, \dots, \widehat{X_p}, \dots, X_{k+1}).
\end{aligned}
$$

Rearranging the arguments in these two sums so as to put X_p into its original position, we see that they exactly cancel the sum in (14.30). This completes the proof that $D\omega$ is multilinear over $C^\infty(M)$, so it defines a smooth $(k+1)$-tensor field.

Since both $D\omega$ and $d\omega$ are smooth tensor fields, we can verify the equation $D\omega = d\omega$ in any frame that is convenient. By multilinearity, it suffices to show that both sides give the same result when applied to an arbitrary sequence of basis vectors in some chosen local frame in a neighborhood of each point. The computations are greatly simplified by working in a coordinate frame, for which all the Lie brackets vanish. Thus, let $(U, (x^i))$ be an arbitrary smooth chart on M. Because both $d\omega$ and $D\omega$ depend linearly on ω, we may assume that $\omega = u\, dx^I$ for some smooth function u and some increasing multi-index $I = (i_1, \dots, i_k)$, so

$$
d\omega = du \wedge dx^I = \sum_m \frac{\partial u}{\partial x^m} dx^m \wedge dx^I.
$$

If $J = (j_1, \dots, j_{k+1})$ is any multi-index of length $k+1$, it follows that

$$
d\omega\left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_{k+1}}}\right) = \sum_m \frac{\partial u}{\partial x^m} \delta_J^{mI}.
$$

The only terms in this sum that can possibly be nonzero are those for which m is equal to one of the indices in J, say $m = j_p$. In this case, it is easy to check that $\delta_J^{mI} = (-1)^{p-1}\delta_{\widehat{J}_p}^I$, where $\widehat{J}_p = (j_1, \dots, \widehat{j_p}, \dots, j_{k+1})$, so

$$
d\omega\left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_{k+1}}}\right) = \sum_{1 \le p \le k+1} (-1)^{p-1} \frac{\partial u}{\partial x^{j_p}} \delta_{\widehat{J}_p}^I. \qquad (14.31)
$$

On the other hand, because all the Lie brackets are zero, we have

$$
\begin{aligned}
&D\omega\left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_{k+1}}}\right) \\
&\quad = \sum_{1 \le p \le k+1} (-1)^{p-1} \frac{\partial}{\partial x^{j_p}} \left(u\, dx^I\left(\frac{\partial}{\partial x^{j_1}}, \dots, \widehat{\frac{\partial}{\partial x^{j_p}}}, \dots, \frac{\partial}{\partial x^{j_{k+1}}}\right)\right) \\
&\quad = \sum_{1 \le p \le k+1} (-1)^{p-1} \frac{\partial u}{\partial x^{j_p}} \delta_{\widehat{J}_p}^I,
\end{aligned}
$$

which agrees with (14.31). □

1. 在$[e'_1, \cdots, e'_n] \sim [e_1, \cdots, e_n]$中，证明$\sim$满足等价关系的三个条件。

自反: $e = eI$, $\det(I) = 1 > 0$

传递: $e'' = e'A$, $e' = eB \Rightarrow e'' = eBA$, $\det(BA) = \det(B)\det(A) > 0$

对称: $e' = eA \Rightarrow e = e'A^{-1}$, $\det(A^{-1}) > 0$

3. 不连通的可定向流形恰有两个定向吗? 否，有 $2^{\text{连通分支个数}}$ 个定向

4. 证明C^∞流形的可定向(或不可定向)的性质在C^∞微分同胚下是不变的。

$\varphi: M^n \to N^n$ 微分同胚，N 可定向，$\exists$ N 上处处非零的 n-form ω

$\omega = g\, dy^1 \wedge \cdots \wedge dy^n$ ， $\varphi^*\omega = g \circ \varphi \cdot \det\left(\frac{\partial y^j}{\partial x^i}\right) dx^1 \wedge \cdots \wedge dx^n$

是 M 上处处非零的 n-form，故 M 可定向

5. 设$(M_1, \mathscr{D}_1)$和$(M_2, \mathscr{D}_2)$都是n维C^∞流形，$F: M_1 \to M_2$是C^∞微分同胚。$\{X_i\}$和$\{X'_i\}$都是$T_p(M_1)$的基，且

$$\overrightarrow{[X_1, \cdots, X_n]} = \overrightarrow{[X'_1, \cdots, X'_n]},$$

证明 $\overrightarrow{[F_{*p}X_1, \cdots, F_{*p}X_n]} = \overrightarrow{[F_{*p}X'_1, \cdots, F_{*p}X'_n]}$。

设 $X_i' = a_i^j X_j$, $F_* X_i' = a_i^j F_* X_j$, $\overrightarrow{[X_1 \cdots X_n]} = \overrightarrow{[X_1' \cdots X_n']}$

$\Rightarrow \det(a_i^j) > 0 \Rightarrow \overrightarrow{[F_* X_1 \cdots F_* X_n]} = \overrightarrow{[F_* X_1' \cdots F_* X_n']}$

7. 设$\{X_i(t)\}$是$T_p(M)$的基，$X_i(t)$关于t连续$(0 \leq t \leq 1)$，证明

$$\overrightarrow{[X_1(0), \cdots, X_n(0)]} = \overrightarrow{[X_1(1), \cdots, X_n(1)]}.$$

设 $X_i(t) = f_i^j(t) \frac{\partial}{\partial x^j}$, $F(t) = (f_i^j(t)) \in GL(n, \mathbb{R})$

$[0,1]$ 连通 $\Rightarrow$ $F(0), F(1)$ 在 $GL(n, \mathbb{R})$ 中处于同一连通分支

$\Rightarrow \det(F(0) \circ F(1)^{-1}) > 0 \Rightarrow \overrightarrow{[X_1(0) \cdots X_n(0)]} = \overrightarrow{[X_1(1) \cdots X_n(1)]}$

8. 设F_i: $U \to R$是C^∞的，其中U是R^n中的开集，且

$$\operatorname{rank}\begin{pmatrix} \frac{\partial F_1}{\partial x^1} & \cdots & \frac{\partial F_1}{\partial x^n} \\ \cdots & \cdots & \cdots \\ \frac{\partial F_{n-k}}{\partial x^1} & \cdots & \frac{\partial F_{n-k}}{\partial x^n} \end{pmatrix} = n-k$$

从第二章2.3例5可知，由方程组

$$\begin{cases} F_1(x^1, \cdots, x^n) = c^1, \\ \cdots\cdots\cdots\cdots \\ F_{n-k}(x^1, \cdots, x^n) = c^{n-k}, \end{cases} \quad (c^i \text{是常数})$$

确定了一个k维C^∞流形M，证明M是可定向的。

$\omega(\lambda_1^i \frac{\partial}{\partial x^i}, \cdots \lambda_k^i \frac{\partial}{\partial x^i})$

$= dx^1 \wedge \cdots dx^n (\frac{\partial F_1}{\partial x^i} \frac{\partial}{\partial x^i}, \cdots \frac{\partial F_{n-k}}{\partial x^i} \frac{\partial}{\partial x^i}, \lambda_i^1 \frac{\partial}{\partial x^i}, \cdots \lambda_i^k \frac{\partial}{\partial x^i})$

$= \det \begin{vmatrix} \partial_1 F_1 & \cdots & \partial_n F_1 \\ \partial_1 F_{n-k} & \cdots & \partial_n F_{n-k} \\ \lambda_1^1 & \cdots & \lambda_n^1 \\ \lambda_1^k & \cdots & \lambda_n^k \end{vmatrix}$

若 $X^j = \lambda_i^j \frac{\partial}{\partial x^i} \in TM$ 且线性无关，则该行列式非零